On the Inconsistency of Classical Logic

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Abstract: This is well-known fact that the classical propositional calculus (zero-order logic, classical propositional logic), is the most fundamental two-valued logical system. This is required for construction of the classical calculus of quantifiers (classical calculus of predicates, first-order logic), which is necessary to construct the classical functional calculus. This last one is needed to formalize the Arithmetic System. At the beginning, we introduce a notation and we repeat some well-known notions (among others, the notions of: operation of consequence, a system, consistency in the traditional sense, consistency in the absolute sense). Next, we present the theorem saying that classical propositional calculus is an inconsistent theory.

Key words: classical propositional calculus, consistency in the traditional sense, consistency in the absolute sense

1. Introduction

The symbols: $\rightarrow$, $\sim$, $\vee$, $\wedge$, $\equiv$ denote the connectives of implication, negation, disjunction, conjunction and equivalence, respectively. $\mathcal{N} = \{1,2,\ldots\}$ denotes the set of all natural numbers.

Next, $At_0 = \{p_1^1, p_2^1, \ldots, p_1^2, p_2^2, \ldots, p_1^k, p_2^k, \ldots\}$ $(k \in \mathcal{N})$ denotes the set of all propositional variables. The symbol S_0 denotes the set of all well-formed formulas, which are built in the usual manner from propositional variables by means of logical connectives. Next, $P_0(\phi)$ denotes the set of all propositional variables occurring in ϕ $(\phi \in S_0)$.

R_{S_0} denotes the set of all rules over S_0. $E(\mathfrak{M})$ is the set of all formulas valid in the matrix $\mathfrak{M}$. The symbol $\mathfrak{M}_2$ denotes the classical two-valued matrix and Z_2 is the set of all formulas valid in the matrix $\mathfrak{M}_2$ (see [10], cf. [1 - 7], [11 - 13]). The symbols $\Rightarrow, \neg, \mathbb{V}, \&, \Leftrightarrow, \forall, \exists$ are metalogical symbols.

Next, $S_0^0 = \{\phi \in S_0 \colon \phi \notin Z_2 \ \& \ \sim\phi \notin Z_2\}$.

Next, r_0 is the symbol of Modus Ponens in propositional calculus. Hence, $R_0 = \{r_0\}$. The formula $X \subset Y$ denotes that $X \subseteq Y$ and $X \neq Y$. For any $X \subseteq S_0$ and $R \subseteq R_{S_0}$, $Cn(R,X)$ is the smallest subset of S_0, containing X, and closed under the rules belonging to R, where $R \subseteq R_{S_0}$.

The couple $\langle R, X\rangle$ is called as a system, whenever $R \subseteq R_{S_0}$, and $X \subseteq S_0$. Hence, $\langle R_0, Z_2\rangle$ denotes the system of the classical propositional calculus.

Now we repeat some well-known definitions (see [10], cf. [5, 7 – 9, 11]). Let $R \subseteq R_{S_0}$ and $X \subseteq S_0$. Then:

Definition 1.1. $\langle R, X\rangle \in Cns^T \Leftrightarrow (\neg\exists\alpha \in S_0)\ [\alpha \in Cn(R,X) \ \& \ \sim\alpha \in Cn(R,X)]$.

Definition 1.2. $\langle R, X\rangle \in Cns^A \Leftrightarrow Cn(R,X) \neq S_0$.

$\langle R, X\rangle \in Cns^T$ denotes that the system $\langle R, X\rangle$ is consistent in the traditional sense. $\langle R, X\rangle \in Cns^A$ denotes that the system $\langle R, X\rangle$ is consistent in the absolute sense (see [10], cf. [11]).

2. The Main Result

Theorem. $\langle R_0, Z_2\rangle \notin Cns^T$. (see [15], cf. [14]).

Proof. Elementary.

□

References

[1] Andrews P. B. *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof*. Applied Logic Series, Vol. 27, Springer Science + Business Media Dordrecht 2002.

[2] Ben-Ari M. *Mathematical Logic for Computer Science*. Springer-Verlag, London 2012.

[3] van Dalen D. *Logic and Structure*. Springer-Verlag, Berlin Heidelberg 2008.

[4] Ershov Y. L. and Palyutin E. A. *Mathematical Logic*. Translated by Shokurov V. Mir Publishers, Moscow 1984.

[5] Grzegorczyk A. *An Outline of Mathematical Logic: Fundamental Results and Notions Explained with All Details*. Translated by Wojtasiewicz O. and Zawadowski W. Synthese Library, Vol. 70, D. Reidel Publishing Company, Dordrecht-Holland/Boston-USA, PWN, Warszawa 1974.

[6] Lassaigne R. and de Rougemont M. *Logic and Complexity*. Springer-Verlag, London 2004.

[7] Marciszewski W. "Sentence Logic." In: *Dictionary of Logic as Applied in the Study of Language. Concepts/Methods/Theories*, edited by Marciszewski W., 334 – 342. *Nijhoff International Philosophy Series*, Vol. 9, Springer Science + Business Media Dordrecht 1981.

[8] Nickles T. "From Copernicus to Ptolemy: Inconsistency and Method". In: *Inconsistency in Science*, Meheus J. (Ed.), 1 – 33. ORIGINS: Studies in the sources of scientific creativity, Vol. 2, Springer Science + Business Media Dordrecht 2002.

[9] Perzanowski J. "Parainconsistency, or Inconsistency Tamed, Investigated and Exploited.", *Logic and Logical Philosophy*, Vol. 9, 5-24 (2001).

[10] Pogorzelski W. A. *The Classical Propositional Calculus*. PWN, Warszawa 1975.

[11] Pogorzelski W. A. and Wojtylak P. *Completeness Theory for Propositional Logics*. Birkhäuser-Verlag AG, Basel Boston Berlin 2008.

[12] Rautenberg W. *A Concise Introduction to Mathematical Logic*. Springer Science + Business Media, LLC 2010.

[13] Srivastava S. M. *A Course on Mathematical Logic.* Springer Science + Business Media, LLC 2008.

[14] Stępień T. J. and Stępień Ł. T. "Theorem on Inconsistency of the Classical Logic." In: *Conference Proceedings, Paris France Sep 21-22, 2017*, 19(9) Part XIII, p.1558, World Academy of Science, Engineering and Technology (2017).

[15] Stępień T. J. and Stępień Ł. T. "On the Inconsistency of Classical Propositional Calculus", *Journal of Mathematics and System Science*, vol. 10, No. 1, 13 - 14 (2020).