On the Inconsistency of: The Classical Propositional Calculus and Its Metatheory

Lukasz Stepien *

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Article

On the Inconsistency of: The Classical Propositional Calculus and Its Metatheory

Ł. T. Stępień

University of the National Education Commission, ul. Podchorazych 2, 30 - 084 Krakow, Poland

Abstract: The classical propositional calculus (zero-order logic, classical propositional logic), is the most fundamental two-valued logical system. In this paper we present a proof of inconsistency of the classical propositional calculus. Then, we get right away the conclusion that the metatheory of the classical propositional calculus is inconsistent.

Keywords: classical propositional calculus; zero-order logic; zeroth-order logic; consistency in the traditional sense; consistency in the absolute sense; inconsistency; metalogic; metatheory1 Introduction

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The issue, whether a given formal system is consistent, is the most fundamental issue for such system.

Many people have been dealing with different aspects of consistency and/or inconsistency in and/or of formal logical systems or in and/or of mathematics or other sciences for e.g. 2-34,36,37,40-51,55,57-63,65-67,70-83,87,88,91,93,94,98,102,105-107,109-113,115.

Probably, the most known example of a system, which inconsistency was (correctly) proved, is Frege's system presented in II volume of his "Grundgesetze der Arithmetik". This inconsistency was proved by Russell in 1903 68 (cf. 13).

The classical propositional calculus is necessary to construct the classical calculus of quantifiers (classical calculus of predicates, first-order logic), and this last one is necessary to construct the classical functional calculus. Classical functional calculus is needed to formalize the Arithmetic System.

So, the significance of the issue of consistency or inconsistency of the classical propositional calculus, is obvious.

One can also consider an impact of inconsistency (in a broad sense), not only on logical systems or on the branches of mathematics, but also on philosophy or semantics 81, and on some applications of logic in computer science 44, functionality of mind 52,56 or psychology 95.

In 38 classical inconsistency of the best-known quantum logic (Birkhoff-von Neumann quantum logic), was discussed.

In 49 Goddard claimed he had proved inconsistency of traditional logic (cf. 50). However, as he wrote this in his paper, in order to prove inconsistency, he considered there an extension of Aristotelian logic, by using negative terms, complex terms, quantified predicates, a theory of obversion etc. In contrary to him, we prove here inconsistency of pure classical propositional calculus (the details are given beneath).

In 2010 Voevodsky delivered a talk entitled "What if current foundations of mathematics are inconsistent?" 108, where he focused on the issue of probably inconsistency of first-order Arithmetic System. In 2011 Nelson claimed he had proved inconsistency of the Arithmetic System, 26,76. However, soon Tao and Tausk found independently an error in Nelson's proof mentioned above, 26.

Ł. T. Stępień, University of the National Education Commission, Kraków, Poland. E-mail: sfstepie@cyf-kr.edu.pl, lukasz.stepien@up.krakow.pl, URL: ltstepien.up.krakow.pl

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Our paper concerns a more fundamental issue, namely inconsistency of the classical propositional calculus. The aim of this paper is to present a proof of the theorem that the classical propositional calculus (the zero-order logic, the classical propositional logic), is inconsistent in the traditional sense and in the absolute sense.

This paper is organized as follows. In the section 2, we introduce a notation and we repeat certain well-known notions (among others, the notions: operation of consequence, a system, consistency in the traditional sense, consistency in the absolute sense) and some well-known theorems. In the next section, we prove some Lemma (Lemma 3.1). The section 4 includes a proof of inconsistency of classical propositional calculus (this result was announced in 99,100,101). The section 5 is devoted to some conclusions.

2. Preliminaries

The symbols: $\rightarrow$, $\sim$, $\vee$, $\wedge$, $\equiv$ denote the connectives of: implication, negation, disjunction, conjunction and equivalence, respectively. $\mathcal{N} = \{1,2,\ldots\}$ denotes the set of all natural numbers.

Next, $At_0 = \{p, q, r, \ldots p_1^1, p_2^1, \ldots, p_1^2, p_2^2, \ldots, p_1^k, p_2^k, \ldots\}$, (where $k \in \mathcal{N}$), denotes the set of all propositional variables. The symbol S_0 denotes the set of all well-formed formulas, which are built in the usual manner from propositional variables by means of logical connectives. We denote the well-formed formulas by small Greek letters (the subscripts, superscripts and/or accents can also be used).

So, $S_0 = \left\{\alpha, \beta, \gamma, \delta, \ldots, \varphi \ldots, \alpha_0, \overset{00}{\delta}, \ldots\right\}$.

R_{S_0} denotes the set of all rules over S_0. $E(\mathfrak{M})$ is the set of all formulas valid in the matrix $\mathfrak{M}$. The symbol $\mathfrak{M}_2$ denotes the classical two-valued matrix and Z_2 is the set of all formulas valid in the matrix $\mathfrak{M}_2$ (see 84, cf. 1,10,35,39,53,64,68,85,92,114).

The symbols $\Rightarrow, \neg, \mathbb{V}, \&, \Leftrightarrow, \forall, \exists$ are metalogical symbols (they denote correspondingly: metaimplication, metanegation, metadisjunction, metaconjunction, metaequivalence and the metalogical quantifiers: general and existential one).

Next, r_0 is the symbol of Modus Ponens in the classical propositional calculus. Hence, $R_0 = \{r_0\}$. The formula $X \subset Y$ denotes that $X \subseteq Y$ and $X \neq Y$. For any $X \subseteq S_0$ and $R \subseteq R_{S_0}$, $Cn(R, X)$ is the smallest subset of S_0, containing X, and closed under the rules belonging to R, where $R \subseteq R_{S_0}$.

The couple $\langle R, X\rangle$ is called as a system, whenever $R \subseteq R_{S_0}$, and $X \subseteq S_0$. Thus, $\langle R_0, Z_2\rangle$ denotes the system of the classical propositional calculus (see 84,85).

Now we repeat some well-known properties of operation of consequence and some well-known definitions (see 84, cf. 1,85,114). Let $R \subseteq R_{S_0}$ and $X \subseteq S_0$. Then:

a_1) $X \subseteq Cn(R, X)$,

a_2) $X \subseteq Y \Rightarrow Cn(R, X) \subseteq Cn(R, Y)$,

a_3) $R \subseteq R' \Rightarrow Cn(R, X) \subseteq Cn(R', X)$,

a_4) $Cn\big(R, Cn(R, X)\big) \subseteq Cn(R, X)$,

a_5) $Cn(R, X) = \bigcup\{Cn(R, Y) : Y \subseteq X \ \&\ \overline{\overline{Y}} < \aleph_0\}$.

Definition 1.1. $\langle R, X\rangle \in Cns^T \Leftrightarrow (\neg\exists\alpha \in S_0)[\alpha \in Cn(R, X) \ \&\ \sim\alpha \in Cn(R, X)]$.

Definition 1.2. $\langle R, X\rangle \in Cns^A \Leftrightarrow Cn(R, X) \neq S_0$.

$\langle R, X\rangle \in Cns^T$ denotes that the system $\langle R, X\rangle$ is consistent in the traditional sense. $\langle R, X\rangle \in Cns^A$ denotes that the system $\langle R, X\rangle$ is consistent in the absolute sense or in Post's sense (see 84, cf. 85,86,114).

Now we repeat some well-known basic Theorems, the so-called metatheorems. The first one is the so-called Deduction Theorem, sometimes called also as Tarski-Herbrand Theorem (see 84, cf. 16,54,85,104):

Theorem 1.1. $(\forall\alpha \in S_0)(\forall\beta \in S_0)(\forall X \subseteq S_0)$

$$[\beta \in Cn(R_0, Z_2 \cup X \cup \{\alpha\}) \Rightarrow (\alpha \rightarrow \beta) \in Cn(R_0, Z_2 \cup X)].$$

The two next metatheorems are correspondingly, the so-called Theorem on Consistency and Theorem on Inconsistency (see 84, cf. 16,54,85,89,104):

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Theorem 1.2.$(\forall\alpha\in S_0)(\forall X\subseteq S_0)[Cn(R_0,Z_2\cup X\cup\{\sim\alpha\})\neq S_0\Leftrightarrow\alpha\notin Cn(R_0,Z_2\cup X)]$.

Theorem 1.3.$(\forall\alpha\in S_0)(\forall X\subseteq S_0)[Cn(R_0,Z_2\cup X\cup\{\alpha\})=S_0\Leftrightarrow\sim\alpha\in Cn(R_0,Z_2\cup X)]$.

At the end of this section we repeat the well-known theorems on consistency of the classical propositional calculus (see 84, cf. 85):

Theorem 1.4. $\langle R_0,Z_2\rangle\in Cns^T$.

Theorem 1.5. $\langle R_0,Z_2\rangle\in Cns^A$.

3. A Lemma

Lemma 3.1.

$(\forall\alpha_0\in A_1')\left(\forall\overset{00}{\delta}\in S_0\right)(\forall\delta\in S_0)$
$(\forall\varphi\in S_0)[Cn(R_0,Z_2\cup A^*\cup A^{**}\cup$
$\left\{\alpha_0\to\left(\overset{00}{\delta}\to\sim\delta\right)\right\}\cup\{\sim\overset{00}{\delta}\to\sim\varphi\})=S_0]$,

where

$A^*=\left\{\alpha_0\to\left(\sim\overset{00}{\delta}\to\varphi\right)\right\}$,

$A^{**}=\left\{\overset{00}{\delta}\to\delta\right\}$,

$A_1=Cn(R_0,Z_2\cup A^*\cup A^{**}\cup$
$\left\{\alpha_0\to\left(\overset{00}{\delta}\to\sim\delta\right)\right\}\cup\left\{\sim\overset{00}{\delta}\to\sim\varphi\right\})$,

$A_1'=Cn(R_0,Z_2\cup A^{**}\cup\left\{\alpha_0\to\left(\overset{00}{\delta}\to\sim\delta\right)\right\}\cup$
$\left\{\sim\overset{00}{\delta}\to\sim\varphi\right\})$.

Proof. Let

1) $\neg(\forall\alpha_0\in A_1')\left(\forall\overset{00}{\delta}\in S_0\right)(\forall\delta\in S_0)$
$(\forall\varphi\in S_0)[Cn(R_0,Z_2\cup A^*\cup A^{**}\cup$
$\left\{\alpha_0\to\left(\overset{00}{\delta}\to\sim\delta\right)\right\}\cup\{\sim\overset{00}{\delta}\to\sim\varphi\})=S_0]$,

where

2) $A^*=\left\{\alpha_0\to\left(\sim\overset{00}{\delta}\to\varphi\right)\right\}$,

3) $A^{**}=\left\{\overset{00}{\delta}\to\delta\right\}$,

4) $A_1=Cn(R_0,Z_2\cup A^*\cup A^{**}\cup$
$\left\{\alpha_0\to\left(\overset{00}{\delta}\to\sim\delta\right)\right\}\cup\left\{\sim\overset{00}{\delta}\to\sim\varphi\right\})$,

5) $A_1'=Cn(R_0,Z_2\cup A^{**}\cup\left\{\alpha_0\to\left(\overset{00}{\delta}\to\sim\delta\right)\right\}\cup$
$\left\{\sim\overset{00}{\delta}\to\sim\varphi\right\})$.

From 1) – 5), we get

6) $(\exists\alpha_0'\in A_1')\left(\exists\overset{00}{\delta}{}'\in S_0\right)(\exists\delta'\in S_0)$
$(\exists\varphi'\in S_0)[Cn(R_0,Z_2\cup A^*\cup A^{**}\cup$
$\left\{\alpha_0'\to\left(\overset{00}{\delta}{}'\to\sim\delta'\right)\right\}\cup\{\sim\overset{00}{\delta}{}'\to\sim\varphi'\})\neq S_0]$,

7) $A^*=\left\{\alpha_0'\to\left(\sim\overset{00}{\delta}{}'\to\varphi'\right)\right\}$,

8) $A^{**}=\left\{\overset{00}{\delta}{}'\to\delta'\right\}$,

9) $A_1=Cn(R_0,Z_2\cup A^*\cup A^{**}\cup$
$\left\{\alpha_0'\to\left(\overset{00}{\delta}{}'\to\sim\delta'\right)\right\}\cup\left\{\sim\overset{00}{\delta}{}'\to\sim\varphi'\right\})$,

10) $A_1'=Cn(R_0,Z_2\cup A^{**}\cup\left\{\alpha_0'\to\left(\overset{00}{\delta}{}'\to\sim\delta'\right)\right\}\cup$

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$\left\{\sim\overset{00}{\delta}{}' \to \sim\varphi'\right\})$.

From 6) – 10), we obtain

11) $(\exists\alpha_0' \in A_1')\left(\exists\overset{00}{\delta}{}' \in S_0\right)(\exists\delta' \in S_0)$

$(\exists\varphi' \in S_0)[\sim\overset{00}{\delta}{}' \to \varphi', \overset{00}{\delta}{}' \to \delta', \overset{00}{\delta}{}' \to \sim\delta',$

$\varphi' \to \overset{00}{\delta}{}', \overset{00}{\delta}{}', \delta', \sim\delta' \in A_1 \;\&\; A_1 = S_0]$,

where

12) $A^* = \left\{\alpha_0' \to \left(\sim\overset{00}{\delta}{}' \to \varphi'\right)\right\}$,

13) $A^{**} = \left\{\overset{00}{\delta}{}' \to \delta'\right\}$,

14) $A_1 = Cn(R_0, Z_2 \cup A^* \cup A^{**} \cup$

$\left\{\alpha_0' \to \left(\overset{00}{\delta}{}' \to \sim\delta'\right)\right\} \cup \left\{\sim\overset{00}{\delta}{}' \to \sim\varphi'\right\})$,

15) $A_1' = Cn(R_0, Z_2 \cup A^{**} \cup \left\{\alpha_0' \to \left(\overset{00}{\delta}{}' \to \sim\delta'\right)\right\} \cup$

$\left\{\sim\overset{00}{\delta}{}' \to \sim\varphi'\right\})$,

what contradicts the steps 6) – 10).

□

4. The Main Result

Theorem 4.1.: $\langle \boldsymbol{R_0}, \boldsymbol{Z_2}\rangle \notin \boldsymbol{Cns^A}$.

Proof.

Let

I) $\langle R_0, Z_2\rangle \in Cns^A$.

By Lemma 3.1, we have

II) $(\forall\alpha_0 \in A_1')\left(\forall\,\overset{00}{\delta} \in S_0\right)(\forall\delta \in S_0)$

$(\forall\varphi \in S_0)[Cn(R_0, Z_2 \cup A^* \cup A^{**} \cup$

$\left\{\alpha_0 \to \left(\overset{00}{\delta} \to \sim\delta\right)\right\} \cup \{\sim\overset{00}{\delta} \to \sim\varphi\}) = S_0]$,

where

III) $A^* = \left\{\alpha_0 \to \left(\sim\overset{00}{\delta} \to \varphi\right)\right\}$

IV) $A^{**} = \left\{\overset{00}{\delta} \to \delta\right\}$

V) $A_1 = Cn(R_0, Z_2 \cup A^* \cup A^{**} \cup$

$\left\{\alpha_0 \to \left(\overset{00}{\delta} \to \sim\delta\right)\right\} \cup \{\sim\overset{00}{\delta} \to \sim\varphi\})$

VI) $A_1' = Cn(R_0, Z_2 \cup A^{**} \cup$

$\left\{\alpha_0 \to \left(\overset{00}{\delta} \to \sim\delta\right)\right\} \cup \{\sim\overset{00}{\delta} \to \sim\varphi\})$.

Hence, by Theorem 1.3, we obtain

VII) $(\forall\alpha_0 \in A_1')\left(\forall\,\overset{00}{\delta} \in S_0\right)(\forall\delta \in S_0)$

$(\forall\varphi \in S_0)\left[\alpha_0, \sim\overset{00}{\delta}, \sim\varphi \in\; Cn\left(R_0, Z_2 \cup A^{**} \cup\; \left\{\alpha_0 \to \left(\overset{00}{\delta} \to \sim\delta\right)\right\} \cup \left\{\sim\overset{00}{\delta} \to \sim\varphi\right\}\right)\right]$,

where

VIII) $A^{**} = \left\{\overset{00}{\delta} \to \delta\right\}$

IX) $A_1' = Cn(R_0, Z_2 \cup A^{**} \cup$

$\left\{\alpha_0 \to \left(\overset{00}{\delta} \to \sim\delta\right)\right\} \cup \{\sim\overset{00}{\delta} \to \sim\varphi\})$.

From VII) – IX), by Duns-Scottus law (Ex Falso Quodlibet), we get

X) $(\forall\alpha_0 \in A_1')[A_1' = S_0]$,

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where
XI) $A^{**} = \{\alpha_0 \to \alpha_0\}$
XII) $A_1' = Cn(R_0, Z_2 \cup A^{**} \cup$
$\{\alpha_0 \to (\alpha_0 \to \sim\alpha_0)\} \cup \{\sim\alpha_0 \to \sim\alpha_0\})$,
and where
XIII) $\overset{00}{\delta} \in \{\alpha_0\}$
XIV) $\delta \in \{\alpha_0\}$
XV) $\varphi \in \{\alpha_0\}$.
Hence, we get
XVI) $(\forall\alpha_0 \in S_0)$
$[Cn(R_0, Z_2 \cup \{\alpha_0 \to (\alpha_0 \to \sim\alpha_0)\}) = S_0]$.
Then, from XVI), by Theorem 1.3. we obtain
XVII) $(\forall\alpha_0 \in S_0)[\alpha_0 \in Cn(R_0, Z_2)]$.
Hence, by Definition 1.1, we have
XVIII) $\langle R_0, Z_2\rangle \notin Cns^T$,
when
XIX) $\alpha_0 \in \{p \wedge \sim p\}$.
Then, by Duns-Scottus law (Ex Falso Quodlibet), and from the fact that $Cn(R_0, Z_2) \subseteq S_0$, we have
XX) $Cn(R_0, Z_2) = S_0$.
Hence, from Definition 1.2, we get
XXI) $\langle R_0, Z_2\rangle \notin Cns^A$,
what contradicts the step I). $\square$

5. Conclusions

If we formulate certain analogon of Definition 1.1. for the case of metatheory of the classical propositional calculus, then from Theorem 4.1. and from Theorem 1.5., we get right away the following conclusion (cf. 32):

Theorem 5.1: The metatheory of the classical propositional calculus is inconsistent.

Let's notice that if one uses only the truth tables, and checks, whether a given formula is a (contr)tautology, then the classical propositional calculus seems to work properly i.e. there is not any contradiction, at least at first sight. The same situation is, when we obtain new laws of the classical propositional calculus, using only the inference rules and the set of axioms.

There in 111 the question on necessity of assumption of truth tables consistency had been asked, and there appearing of the inconsistency, in the context of the truth tables, was demonstrated (as the Authors of 111 have established there), by using the case of liar paradox. In this paper mentioned above, a construction of truth tables in a consistency-independent paraconsistent setting was presented. The Authors of 111 had been working there just using paraconsistent metatheory. On the other hand, there in 103 were presented some arguments against classical paraconsistent metatheory.

Anyway, we would like to stress here that we have not used any truth tables in this current paper. In the steps XVIII) – XIX) of the proof of the Main Result, we have obtained that the classical propositional calculus is inconsistent in the traditional sense, when $\alpha_0 \in \{p \wedge \sim p\}$ (so, α_0 is some contrtautology), however any liar paradox has not been involved here. We have applied: some laws of the classical propositional calculus, Modus Ponens rule r_0, Theorem 1.3, Definition 1.1. and Definition 1.2.

Some remarks on the case of inconsistent metatheory, are included in 59,88 and 110 (inconsistency of the so-called Nudelman's metatheory, was proved in 45).

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